B.Sc. Part III (Hons) 7th Paper

Diff. equations (contd.)

Standard Form III

Form → $\boxed{f(p,q,z)=0}$

i.e. equation contains p, q and z only; not x and y.

Method Put $u = x + ay$, a is a constant.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}\cdot a = a\frac{dz}{du}$$

Put these values in the given equation.

⇓

resulting is an ordinary diff. eqn of Ist order

↓

Solve it

↓

Put $u = x + ay$

↓

Result = complete integral

Example 1 Solve

$$9(p^2z+q^2)=4$$

Soln The given equation

$$9(p^2z+q^2)=0 \quad \text{---}(1)$$

It contains p, q and z only, i.e. it is of the form $\boxed{f(p,q,z)=0.}$

Put $u = x+ay$.

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{dz}{du} \text{ and } q=\frac{\partial z}{\partial y}=a\frac{dz}{du}$$

so, eq (1) becomes

$$9\left[\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{dv}\right)^2\right]=4$$

$$\Rightarrow 9\left(\frac{dz}{du}\right)^2\left[z+a^2\right]=4$$

$$\Rightarrow 3\frac{dz}{du}=\pm\frac{2}{\sqrt{z+a^2}}$$

$$\Rightarrow \frac{dz}{du}=\pm\frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \int\sqrt{z+a^2}\,dz=\pm\frac{2}{3}du$$

Integrating, we get

$$\Rightarrow \frac{(z+a^2)^{3/2}}{3/2} = \pm \frac{2}{3}(u+k)$$

$$\Rightarrow (z+a^2)^{3/2} = \pm (u+k)$$

$$\Rightarrow (z+a^2)^{3/2} = \pm (x+ay+k) \quad [\because u = x+ay]$$

squaring both sides,

$$\Rightarrow (z+a^2)^3 = (x+ay+k)^2$$

This is the complete integral of the given differential equation.

Ex.2 Find the complete integral of

$$p^2 = qz.$$

Soln The given equation

$$p^2 = qz \quad \text{——— (1)}$$

This is of the form $f(p,q,z)=0$.

Put $u = x+ay$.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{dz}{du} \text{ and } q = \frac{\partial z}{\partial y} = a\frac{dz}{du}.$$

Putting these values in (1), we get

$$\left(\frac{dz}{du}\right)^2 = a\frac{dz}{du}\cdot z$$

$$\Rightarrow \left(\frac{dz}{du} - az\right)\frac{dz}{du} = 0$$

$$\Rightarrow \frac{dz}{du} - az = 0 \quad \Rightarrow \frac{dz}{z} - a\,du = 0$$

integrating, we get

$$\Rightarrow \log z - au = b$$

$$\Rightarrow \log z = au + b = a(x+ay) + b \quad \text{---}(2)$$

$$\Rightarrow z = e^{a(x+ay)} \cdot e^{b}$$

$$\Rightarrow z = k\,e^{a(x+ay)} \quad \text{---}(3)$$ where $k = e^{b}$ is any constant.

Eq (2) or (3) gives the complete integral of the given differential equation.